[07-10-03-T12] REV1

Preliminary for Binomial Thm Proof

Prerequisite

[1] Pascal's Law: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. Equivalent to $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

[2] $\binom{n}{n} = \binom{n}{0} = \binom{0}{0} = \binom{n+1}{n+1} = \binom{n+1}{0} = 1$,

[3] Lemma. $\sum_{k=m}^{n} a_k\, c_k + \sum_{k=m}^{n} b_k\, c_k = \sum_{k=m}^{n} (a_k + b_k)\, c_k$

[4] Combinatorial argument for the coefficients of $(a+b)^n$

■ Pascal's Law. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Proof

$$\text{RHS} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$= \frac{(n-1)!}{(k-1)!\,[(n-1)-(k-1)]!} + \frac{(n-1)!}{k!\,[(n-1)-k]!}$$

$$= \frac{(n-1)!}{(k-1)!\,[n-k]!} + \frac{(n-1)!}{k!\,[n-k-1]!}$$

$$= \frac{k(n-1)!}{k!\,[n-k]!} + \frac{(n-k)\,(n-1)!}{k!\,[n-k]!}$$

$$= \frac{k(n-1)! + (n-k)\,(n-1)!}{k!\,[n-k]!}$$

$$= \frac{(n-1)![k+(n-k)]}{k!\,[n-k]!}$$

$$= \frac{n!}{k!\,[n-k]!}$$

$$= \binom{n}{k}$$

$$= \text{LHS}$$

$\therefore \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

■ Lemma. $\sum_{i=m}^{n} a_i c_i + \sum_{i=m}^{n} b_i c_i = \sum_{i=m}^{n} (a_i + b_i) c_i$

LHS $= \sum a_i c_i + \sum b_i\ c_i$

$= (a_1 c_1 + a_2 c_2 + a_3 c_3 + \ldots + a_n c_n) + (b_1 c_1 + b_2 c_2 + b_3 c_3 + \ldots + b_n c_n)$

$= a_1 c_1 + b_1 c_1 + a_2 c_2 + b_2 c_2 + \ldots + a_n c_n + b_n c_n$

$= c_1(a_1 + b_1) + c_2(a_2 + b_2) + c_3(a_3 + b_3) + \ldots + c_n(a_n + b_n)$

$= \sum (a_i + b_i) c_i$

= RHS

■ Combinatorial argument for coefficients of $(a + b)^n$

■ Make every 6-person collection from 6 bride-groom couples such that no groom appears with his bride.

1st collection:

choose no couple from which to take the groom	$\binom{6}{0}$	ways
choose the 6 brides from the remaining 6 couples	1	way

2nd collection:

choose a couple from which to take the groom	$\binom{6}{1}$	ways
choose the 5 brides from the remaining 5 couples	1	way

3rd collection:

choose 2 couples from which to take the grooms	$\binom{6}{2}$	ways
choose the 4 brides from the remaining couples	1	way

4th collection:

choose 3 couples from which to take the grooms	$\binom{6}{3}$	ways
choose the 3 brides from the remaining couples	1	way

5th collection:

choose 4 couples from which to take the grooms	$\binom{6}{4}$	ways
choose the 2 brides from the remaining couples	1	way

6th collection:

choose 5 couples from which to take the grooms	$\binom{6}{5}$	ways
choose the 1 bride from the remaining couples	1	way

7th collection:

choose 6 couples from which to take the grooms	$\binom{6}{6}$	ways
choose no brides	1	way

■ Less romantic

Make every 6-letter collection from 6 (a, b) pairs such that no a appears with its b.

The solution is the same as for the more romantic version.

■ Finally

Make every 6-letter collection from 6 $(a + b)$ binomials such that no a appears with its b.

The solution is the same as those just considered. We'll just recite the answer:

$a^6 b^0$	$\binom{6}{0}$ ways
$a^5 b^1$	$\binom{6}{1}$ ways
$a^4 b^2$	$\binom{6}{2}$ ways
$a^3 b^3$	$\binom{6}{3}$ ways
$a^2 b^4$	$\binom{6}{4}$ ways
$a^1 b^5$	$\binom{6}{5}$ ways
$a^0 b^6$	$\binom{6}{6}$ ways

Note that the sum of the exponents is in each case 6, since it is 6-letter collections that are desired.

Thus,

$$(a+b)^6 = \binom{6}{0} a^6 b^0 + \binom{6}{1} a^5 b^1 + \binom{6}{2} a^4 b^2 + \binom{6}{3} a^3 b^3 + \binom{6}{4} a^2 b^4 + \binom{6}{5} a^1 b^5 + \binom{6}{6} a^0 b^6$$

To generalize,

$$(a+b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \cdots + \binom{n}{n} a^{n-n} b^n$$

Equivalently,

$$(a+b)^n = \sum_{m=0}^{n} \binom{n}{m} a^{n-m} b^m \tag{1}$$

Equation (1) is known as the binomial theorem.